2101. Prove that $2^m - 1$, where m is an even number, is only prime if $m = 2$.

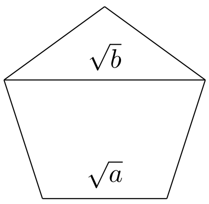
2102. Find the area enclosed by the curves $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.

2103. A particle has position $\mathbf{r} = \begin{pmatrix} 1+t \\ -t^2 \\ 4+2t^2 \end{pmatrix}$ m.

(a) Find the magnitude of the acceleration.

(b) Hence, or otherwise, show that the path of the particle is parabolic.

2104. The diagram shows a regular pentagon of side length $\sqrt{a}$ and diagonal length $\sqrt{b}$.



Show that $a : b$ is $1 : 2(1 - \cos 108°)$.

2105. Express the following set as a list of elements, in the form $\{a_1, a_2, ..., a_n\}$.

$$\{x \in \mathbb{Z} : e^x < 1000\} \cap \{y \in \mathbb{Z} : 2^y > 7\}.$$

2106. Bags of fertiliser are sold with a nominal mass of 1 kg. The true mass M for one bag is modelled as follows: $M \sim N(1.01, 0.0004)$.

(a) Find the probability that

i. a bag of fertiliser is underweight as sold,

ii. at least one of a pair of bags of fertiliser is underweight as sold.

(b) What assumption have you made?

2107. The parabola $y = -x^2 - x + 7$ has a maximum at point (p, q). Find the equation of the monic parabola which has a minimum at point (p, q).

2108. Explain why any contact force exerted on the curved surface of a smooth cylinder must have a line of action that passes through the cylinder's axis of symmetry.

2109. When the eight letters of the word ARRANGED are arranged at random, the probability p that the original word is spelled out is given by

$$p = \frac{2! \times 2!}{8!}.$$

Explain this calculation.

2110. Sketch $y = \sin^2 x$, with x in radians.

2111. Eliminate t from the following equations:

$$p = a + b \sec t,$$
$$q = c + d \tan t.$$

2112. The first three terms of a geometric sequence are $b - 4$, b, $2b + 6$. Find all possible values of the common ratio.

2113. A quintic graph Q has equation

$$y = (x^2 + 1)(3x - 2)^3.$$

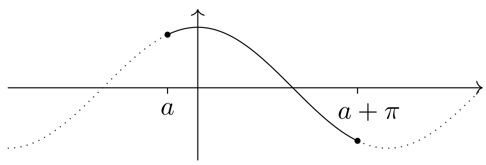
(a) Show that Q is stationary and inflected at its x axis intercept.

(b) Show that Q has no other stationary points.

(c) Hence, sketch Q.

2114. Exactly four forces, of magnitudes 3, 5, 10, and T Newtons, act on an object, which is in equilibrium. Determine the set of possible values of T.

2115. In this question, the cosine function is restricted to the domain $[a, a + \pi]$, where $a \in \mathbb{R}$. The range of this restricted cosine function is the interval $[p, q]$, where p and q depend on a.



Find the greatest and least possible values of $q - p$.

2116. Sketch a counterexample to the following: "If a polynomial function f has at least one root, then it must have at least one fixed point."

2117. A chord is drawn to $y = x^4 - 4x^3 + 5x^2 - x - 1$ at x values 0 and 2.

(a) Find the equation of this chord.

(b) Show that the chord is tangent to the curve.

2118. Find y in simplified terms of x, if

$$\log_2 y - \log_4 y = \log_2 x + \log_4 x.$$

2119. From a population of a hundred, a random sample of five people is taken, and their heights measured. Height is measured to enough accuracy to make the probability of two people having precisely the same height negligible. Find the probability that all five people sampled are above the population median height.

2120. Solve $(\sqrt{x} + x)^3 + (\sqrt{x} - x)^3 = 4\sqrt{x}$.

2121. A student writes: "When you press the accelerator of a resting car, a driving force is generated which acts backwards on the ground." Explain whether this is correct.

2122. A pair of distinct quadratic functions f and g have $\text{f}(a_i) = \text{g}(a_i)$ for $i = 1, 2$, where $a_1 \neq a_2$.

Prove by contradiction that $\text{f}'(a_1) \neq \text{g}'(a_1)$ and $\text{f}'(a_2) \neq \text{g}'(a_2)$.

2123. The growth rates of cultures of a mould are being analysed, when treated with various substances. Usually, for such cultures, growth rate R follows $R \sim N(1.1802, 0.0025)$. Substance P216, whose effect on this mould is unknown, is applied to a sample of twenty-four cultures, and a hypothesis test, at the 1% level, is set up to determine whether this application has an effect on mould growth.

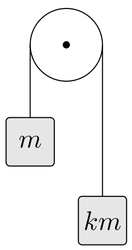
(a) Write down suitable hypotheses for the test, defining any variables you use.

(b) Calculate the critical region for the test.

(c) The test statistic is 1.1624. Complete the test, writing your conclusions in context.

2124. Either prove or disprove the following statement: "For an invertible function g, the functions g and g^{-1} have the same set of fixed points."

2125. A smooth pulley system, in which all elements but the two marked masses are light and $k > 1$ is a constant, is set up as below.



(a) Show that the acceleration of the system may be expressed as

$$a = \left(1 - \frac{2}{k+1}\right) g.$$

(b) What assumption have you made about the string, beyond the fact that it is light?

(c) Describe the behaviour of the system for very large values of k.

2126. The parabola $y = (ax+b)(cx+d)$ has a stationary point on the x axis. Show that $ad - bc = 0$.

2127. Explain why, if events A and B are independent, they can only be mutually exclusive if either $\mathbb{P}(A)$ or $\mathbb{P}(B)$ is zero.

2128. In each case, state the value of the limit:

(a) $\displaystyle\lim_{x\to 0} \frac{2^x}{2^x+1}$,

(b) $\displaystyle\lim_{x\to 1} \frac{2^x}{2^x+1}$,

(c) $\displaystyle\lim_{x\to \infty} \frac{2^x}{2^x+1}$.

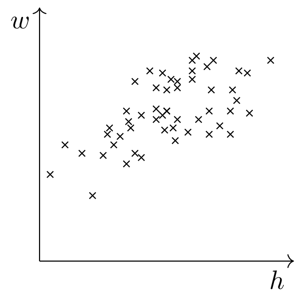
2129. State, with a reason, whether the following holds: "If the function f is increasing at every point in its domain, then, for $a, b \in \mathbb{R}$, $a \leq b \implies \text{f}(a) \leq \text{f}(b)$."

2130. Two operations are defined on real numbers a, b by $a \oplus b = a^2 + b^2$ and $a \otimes b = a^2 b^2$.

(a) Factorise $a^2 \oplus b^2 - 2(a \otimes b)$ fully.

(b) Show that $(a \oplus b) \otimes c \not\equiv (a \otimes c) \oplus (b \otimes c)$.

2131. A census is taken, recording the heights h and weights w of children at a particular school. These are plotted on a scatter diagram as follows, and the correlation coefficient ρ is calculated:



(a) Explain why the symbol ρ has been used for the correlation coefficient, rather than r.

(b) Describe the relationship between w and h.

(c) In fact, a pupil at the 90th percentile for height and the 10th percentile for weight was away on the day of the census. Describe the effect on ρ of reinstating this pupil.

2132. Prove that, for any linear function $\text{f}(x) = ax + b$,

$$\int_0^4 \text{f}(x)\,dx = 2\int_1^3 \text{f}(x)\,dx.$$

2133. Three dice are rolled, and the scores added. Find the probability that the total is at least 17.

2134. A student makes the following claim: "If objects are not accelerating relative to each other, then internal forces between them can be ignored when applying NII". State, with a reason, whether this is correct.

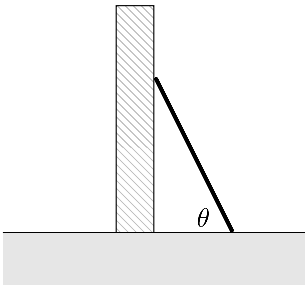
2135. Prove that, for triangular numbers $T_n = \frac{1}{2}n(n+1)$, the following equality holds for all $a, b \in \mathbb{N}$:

$$T_{a+b} = T_a + T_b + ab.$$

2136. Show that the cubic function $\mathrm{f}(x) = a(x-b)^3$, for non-zero constants $a, b \in \mathbb{R}$, is convex when its value is positive and concave when its value is negative.

2137. Either prove or disprove the following statement: "Given two linear equations in two unknowns x and y, the solution set always contains exactly one element."

2138. A ladder of mass m kg and length 2 m is standing on rough ground, leaning against a smooth wall. The coefficient of friction at the base is $\mu = \frac{\sqrt{3}}{6}$. The base of the ladder is placed as far away from the wall as possible while maintaining equilibrium. In this position, the ladder makes an angle θ with the horizontal.



(a) Explain why the friction acting on the ladder has magnitude

$$F = \tfrac{\sqrt{3}}{6} R_{\text{floor}}.$$

(b) Set up equations for vertical and horizontal equilibrium, and moments around the base of the ladder.

(c) Hence, determine the value of θ.

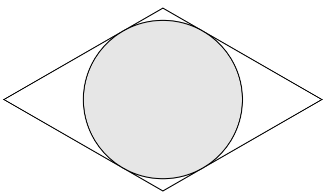
2139. Explain why this seemingly nonsensical equation is, in fact, less impossible than it looks:

$$\int \mathrm{f}(x)\,dx = 1 + \int \mathrm{f}(x)\,dx.$$

2140. State, with a reason, whether it is true that

$$y = \mathrm{f}(1-x) \implies \frac{dy}{dx} = \mathrm{f}'(1-x).$$

2141. The largest possible circle is constructed inside a rhombus whose interior angles are 60° and 120°.



Show that the ratio of the areas of the two shapes is $8\sqrt{3} : 3\pi$.

2142. Find constants P, Q to make the below an identity:

$$\frac{1}{(x^2-1)^2} \equiv \frac{P}{(x-1)^2} + \frac{Q}{(x+1)^2}.$$

2143. Triangle T has sides length $n, n+1, n+2$, where $n > 3$. Using the cosine rule, prove that T is an acute triangle.

2144. The cubic graph $y = \mathrm{f}(x)$ has a stationary point of inflection at $(0, 4)$ and passes through the point $(2, 28)$.

(a) Explain why f must be expressible in the form $\mathrm{f}(x) = ax^3 + 4$, for $a \in \mathbb{R}$.

(b) Hence, determine the equation of the graph.

2145. Determine the number of roots of the equation

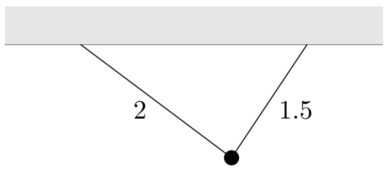
$$\sin^2 x + \sin x - 12 = 0.$$

2146. The function g is defined over the reals, and has range $(-k, k)$. Give the ranges of the following:

(a) $x \mapsto \dfrac{1}{k^3}\left[\mathrm{g}(x)\right]^3$,

(b) $x \mapsto \dfrac{1}{k^4}\left[\mathrm{g}(x)\right]^4$.

2147. A 20 kg lighting rig is hung from two ropes, whose lengths are 1.5 m and 2 m. These are attached to two points, 2.5 m apart horizontally, in the rafters of an auditorium.



Determine the tensions in the two ropes.

2148. State, with a reason, whether the following gives a well-defined function:

$$\mathrm{h}: \begin{cases} \mathbb{Q}^+ \mapsto \mathbb{Q}^+ \\ x \mapsto \sqrt{x}. \end{cases}$$

2149. Show that $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{2}$.

2150. In a random trial, events A and B are such that

$$\mathbb{P}(A \mid B) = \mathbb{P}(A' \mid B) = \mathbb{P}(B).$$

Determine $\mathbb{P}(A \cap B)$.

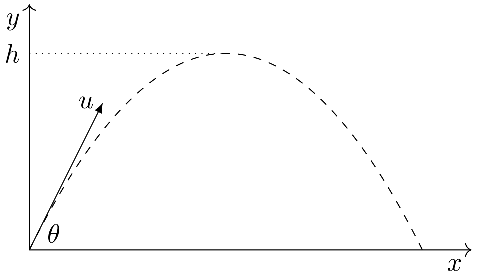
2151. By differentiating from first principles, prove that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

2152. Show that $y = x^4 - x^3 + x^2 - x + 2$ is convex.

2153. Find $\int \frac{1}{\cos^2 5x}\, dx$.

2154. A projectile is fired at speed u, at an angle θ above the horizontal.



Prove that the greatest height attained is

$$h = \frac{u^2 \sin^2\theta}{2g}.$$

2155. The midpoints of the edges of triangle ABC have coordinates $(2,2)$, $(-1,5)$ and $(4,4)$. Determine the coordinates of points A, B, C.

2156. Sketch $y = x^4 + x^3$.

2157. The inequality $\frac{x-2}{x-3} \leq 1$ is given.

(a) Explain why multiplying by the denominator to get $x - 2 \leq x - 3$ would be incorrect.

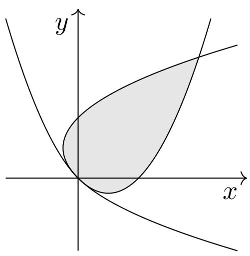
(b) Write down the boundary equation, and show that it has no roots.

(c) The LHS can be rewritten

$$\frac{x-2}{x-3} \equiv a + \frac{b}{x-3}.$$

Use this form to solve the inequality.

2158. The graphs $y = x^2 - x$ and $x = y^2 - y$ are shown:



(a) Find the intersections of the curves.

(b) Show that the shaded region has area $\frac{8}{3}$.

2159. Give a counterexample to the following claim: "If an iteration has no fixed points, then it cannot return to its starting value."

2160. The following differential equation is first-order and *non-separable*:

$$\frac{dy}{dx} + \frac{y}{x} = 3x.$$

Verify that the solution curve through $(1,2)$ is

$$y = x^2 + \frac{1}{x}.$$

2161. It is given that $y = \mathrm{f}(x)$ has the y axis as a line of reflective symmetry, and also that

$$\int_0^1 \mathrm{f}(x)\, dx = k.$$

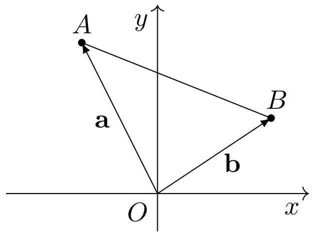
Evaluate the following integrals, in terms of the constant k where necessary:

(a) $\int_{-1}^{1} \mathrm{f}(x)\, dx$,

(b) $\int_0^1 \mathrm{f}(x) - \mathrm{f}(-x)\, dx$.

2162. Sketch any $y = \mathrm{f}(x)$ which is convex on $(-\infty, 0)$, concave on $(0, \infty)$, and passes through O.

2163. The line segment AB has endpoints with position vectors $\mathbf{a}$ and $\mathbf{b}$, relative to the origin O:



Point P is then defined, for positive constants λ_1, λ_2, by the position vector $\mathbf{p} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}$. Prove that, if $\lambda_1 + \lambda_2 = 1$, then P is on line segment AB.

2164. True or false?

(a) $x, y \in [0,1] \implies x + y \in [0,1]$,

(b) $x, y \in [0,1] \implies x - y \in [0,1]$,

(c) $x, y \in [0,1] \implies xy \in [0,1]$.

2165. In a *primitive* Pythagorean triple (a, b, c), the three values $a, b, c \in \mathbb{N}$ share no prime factor. Prove by contradiction that, in such a primitive triple, c cannot be even.

2166. Explain why $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$.

2167. Disprove the following statement:

"The normal to a cubic at its centre of rotational symmetry intersects the curve three times."

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2168. Four independent trials of the same experiment are performed. The probability of success in any one trial p, where $0 < p < 1$, is unknown, giving the distribution $X \sim B(4, p)$. It is observed, over many sets of four trials, that

$$\mathbb{P}(X = 1) \approx \mathbb{P}(X = 2).$$

Determine the approximate value of p.

2169. Simplify $\dfrac{d}{dx}(x+y) \times \dfrac{d}{dy}(x-y)$

2170. The parabola $y = ax^2 + bx + c$ is stationary on the y axis. Explain whether any of the constants a, b, c can be determined from this information.

2171. State, with a reason, whether or not the following are valid identities:

(a) $\operatorname{cosec}|x| \equiv \operatorname{cosec} x$,

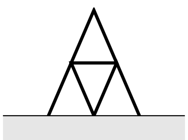
(b) $\sec|x| \equiv \sec x$,

(c) $\cot|x| \equiv \cot x$.

2172. A polynomial function g has the property that, for all $k \in (0, \infty)$, $\mathrm{g}(k) < \mathrm{g}(0)$. Show, by finding a counterexample, that this does not guarantee that $\mathrm{g}'(0) < 0$.

2173. The expression $5x^3+kx^2+kx-1$ has (x^2+x+1) as a factor. Determine the roots of $5x^3+kx^2+kx-1$.

2174. The two-storey house of cards below requires seven identical playing cards.



Show that an n-storey house of cards requires $\frac{1}{2}n(3n+1)$ cards.

2175. Various bivariate data points are given as follows:

x	0	1	2
y	3	4	10

It is proposed that the relationship between x and y be modelled with an exponential curve of the form $y = Ae^{kx}$. Show that this is not suitable.

2176. Show that $\displaystyle\int_0^1 60x\sqrt{1+4x}\,dx = 1 + 25\sqrt{5}$.

2177. For $k \in \mathbb{N}$, and $a \neq b$, write down the equation(s) of any vertical asymptotes of the graph

$$y = \frac{x^{2k+1} - a^{2k+1}}{x^{2k+1} - b^{2k+1}}.$$

2178. Three-tenths of the leaves on a particular tree are infected by leaf-miner moths. Of infected leaves, 60% fall early; of healthy leaves, 20% do so.

(a) Represent this information on a tree diagram.

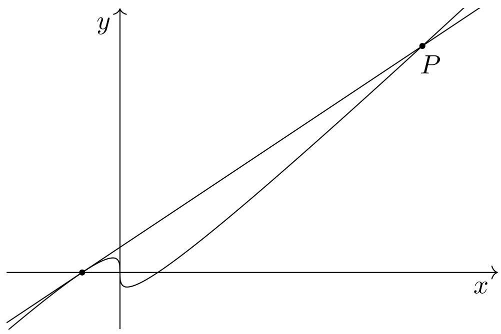
(b) Find the probability a leaf falls early.

(c) Show that, among leaves that fall early, the probability of infection by leaf-miner moths is around 56%.

2179. Shade the region of the (x, y) plane which satisfies both of the following inequalities, for $a, b > 0$:

$$y - b < x - a,$$
$$(x-a)^2 + (y-b)^2 < 1.$$

2180. A tangent is drawn to $y = x - x^{\frac{1}{3}}$ at $x = -1$.



Find the coordinates of point P, where the tangent re-intersects the curve.

2181. Evaluate $\displaystyle\lim_{x\to 4} \frac{x^2 - 16}{x^2 + x - 20}$.

2182. Write the following in terms of e^x:

(a) $\sqrt{e^{3x}}$,

(b) $\sqrt{e^{3x+1}}$,

(c) $\sqrt{e^{3x+2}}$.

2183. Two dice have been rolled, and the scores have been recorded as A and B. Determine whether knowing the fact "A is prime" increases, decreases or doesn't change $\mathbb{P}(|A - B|) = 1$.

2184. You are given that the equation $px^3+qx^2+rx = 0$, where p, q, r are non-zero constants, has exactly two real roots. State, with a reason, whether each of the following equations can be guaranteed to have exactly two real roots:

(a) $(p+1)x^3 + (q+1)x^2 + (r+1)x = 0$,

(b) $p(2x-3)^3 + q(2x-3)^2 + c(2x-3) = 0$,

(c) $px^4 + qx^3 + rx^2 = 0$.

2185. A region of the (x, y) plane is defined by

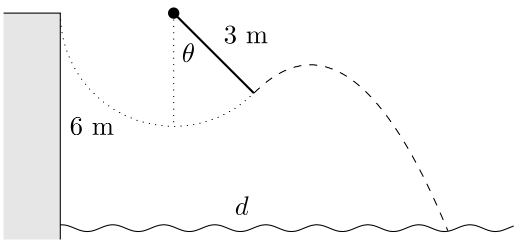
$$0 \le y - x^2 \le 1.$$

Prove that this region has infinite area.

2186. Determine the least n such that the product of n consecutive integers can be guaranteed to end 00.

2187. Take $g = 10$ in this question.

Some children are jumping into a lake, using a rope tied to a tree branch. They step from a rock, swing in a circular arc holding onto the rope, then let go when the rope makes an angle θ with the vertical. The rope is 3 m long, and the rock is 6 m high. The horizontal distance from rock to splashdown is denoted d m.



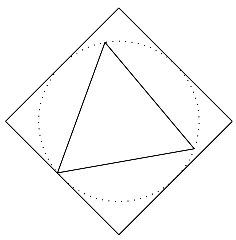
Find the exact value of d when a child releases at

(a) $\theta = 90°$,
(b) $\theta = 0°$, moving at $\sqrt{60}$ ms^{-1},
(c) $\theta = 60°$, moving at $\frac{\sqrt{3}}{3}$ ms^{-1}.

2188. A sample $\{(x_i, y_i)\}$ of bivariate data is taken, and its correlation coefficient is calculated as r. The y values are then coded by $y_i \mapsto ay_i + b$. Explain whether this affects the value of r.

2189. A student says: "To get the graph of an inverse function, we switch the inputs and outputs, so we switch x and y. Hence, the graph of $y = \arcsin x$ is the graph $x = \sin y$." Explain the (small) error.

2190. Let T be the largest equilateral triangle which can rotate freely about a single point while remaining inside a unit square.



Show that the perpendicular height of T is $\frac{3}{4}$.

2191. Show that the value of the function $x \mapsto e^x(1-e^x)$ at its point of inflection is $\frac{3}{16}$.

2192. The question concerns the rate of change of x^3 with respect to x^2. To find this quantity, the variable u is defined as $u = x^2$.

(a) Find $\dfrac{du}{dx}$ in terms of x.
(b) Show that $\dfrac{d}{du}(x^3) = 3x^2 \div \dfrac{du}{dx}$.
(c) Hence, find $\dfrac{d}{d(x^2)}(x^3)$ in terms of x.

2193. Two cards are dealt from a standard pack.

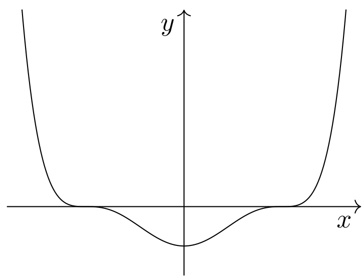
(a) Find
  i. $\mathbb{P}$(same suit),
  ii. $\mathbb{P}$(same number, given same suit).
(b) Hence, determine whether the events "same suit" and "same number" are independent.

2194. Prove, by contradiction, that four distinct points which lie on the parabola $y = x^2$ cannot form a rectangle.

2195. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

① $y = e^x$,
② $x = \ln y$.

2196. Determine whether the equation $y = \left(x^2 + x\right)^3$ could produce the following graph:



2197. An equation is given as

$$\sin^2 2x + \cos 4x = 0.$$

Show that the solution is

$$x = \left(\tfrac{1}{2}n - \tfrac{1}{4}\right)\pi, \text{ for } n \in \mathbb{Z}.$$

2198. A variable is known to have a distribution well modelled by $X \sim B(20, 0.1)$. A statistics student subsequently attempts to approximate this with a normal distribution $Y \sim N\big(\mu, \sigma^2\big)$.

(a) Find the percentage error when this normal approximation is used to find $\mathbb{P}(0 \le X \le 1)$.
(b) Comment on your answer.

2199. In this question, $\text{Int}\big(\text{f}(x)\big)$ is defined as the definite integral of $\text{f}(x)$ between $x = 0$ and $x = 1$.

An equation is given as

$$\text{Int}\left(x^2 - k\right) = \text{Int}\left(x^4\right).$$

Solve for the constant k.

2200. Prove that $\displaystyle\lim_{x\to\infty} \frac{2x+3}{3x+2} = \frac{2}{3}$.

——— End of 22nd Hundred ———